Object Correspondence as a Machine Learning Problem

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In this lecture we

- repeat the main ideas of *Kernel Methods* and *RKHS*
- discuss the role of *registration*
- repeat the elements of the registration framework
- explain Schölkopf’s approach to registration.
RKHS and Kernels

Define a mapping $\phi$ from the **arbitrary** input set $\mathcal{X}$ into the space $\Phi$.

Let $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a kernel.

Define

$$
\phi : \mathcal{X} \to \Phi
x \to K(x, \cdot).
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Main fact!

Every positive definite kernel $K$ uniquely induces an RKHS.
Theorem (Generalized Representer Theorem)

Given a nonempty set $\mathcal{X}$, a pd real-valued kernel $K : \mathcal{X} \times \mathcal{X}$, a training sample $(x_1, y_1), \ldots, (x_m, y_m)$, a strictly monotonically increasing real-valued function $g$ on $[0, \infty[$, an arbitrary loss function $L : (\mathcal{X} \times \mathbb{R}^2)^m \to \mathbb{R} \cup \infty$, and a class of functions

$$\mathcal{H} := \left\{ f \in \mathbb{R}^\mathcal{X} | f(\cdot) = \sum_{i=1}^{\infty} \beta_i K(\cdot, z_i), z_i \in \mathcal{X}, \|f\| < \infty \right\}$$

then any $f \in \mathcal{H}$ minimizing the regularized risk functional

$$L((x_1, y_1, f(x_1)), \ldots, (x_m, y_m, f(x_m))) + g(\|f\|)$$

admits a representation of the form

$$f(\cdot) = \sum_{i=1}^{m} \alpha_i K(\cdot, x_i)$$
Kernels from Feature Maps

**Theorem**

Let

$$\phi : \mathcal{X} \rightarrow \Phi$$

be a feature map and $\Phi$ be an inner product space. Then the function

$$\langle \phi(x), \phi(x') \rangle =: K(x, x')$$

is a positive definite kernel.
Registration as a feature map

- Unregistered objects don’t form a vector space.

\[
\frac{1}{2} \cdot \mathbf{x} + \frac{1}{2} \cdot \mathbf{y} = \mathbf{z}
\]
Registration as a feature map

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\[ \frac{1}{2} \cdot \quad + \quad \frac{1}{2} \cdot \quad = \]

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Registration as a feature map

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  \[ \frac{1}{2} \cdot \text{image} + \frac{1}{2} \cdot \text{image} = \text{image} \]

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Interpretation

A registration algorithm can be seen as a feature map \( \phi \), which turns the unstructured set \( \mathcal{X} \) into a vector space.
Registration as a feature map

- We can represent the object class as an embedding in $\mathbb{R}^N$.
  - every object is written as an $N$-dimensional vector.

$$\Gamma_1 = \begin{bmatrix} x_1^1 \\ y_1^1 \\ z_1^1 \\ x_2^1 \\ y_2^1 \\ z_2^1 \end{bmatrix} \quad \Gamma_i = \begin{bmatrix} x_i^1 \\ y_i^1 \\ z_i^1 \\ x_i^2 \\ y_i^2 \\ z_i^2 \end{bmatrix}$$

- For a good $\phi : \mathcal{X} \rightarrow \mathbb{R}^N$, linear combinations of the objects “make sense”.
- The standard inner product measures similarity.
- $K(\Gamma_1, \Gamma_i) := \langle \phi(\Gamma_1), \phi(\Gamma_i) \rangle_{\mathbb{R}^N}$ is a similarity measure.
The registration framework

All registration algorithms consist of the following parts:

- A distance function $\mathcal{D}$
- A regularization term $\mathcal{R}$.
- A set of admissible transformations $\mathcal{H}$.

Registration as an optimization problem

Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^3$ be two surfaces and $\Phi \in \mathcal{H}$. The registration problem reads:

$$\min_{\Phi \in \mathcal{H}} \mathcal{D}[\Gamma_1, \Gamma_2, \Phi] + \lambda \mathcal{R}[\Phi]$$
Object representation

A mathematically convenient representation of complicated surfaces is the *implicit-surface* or *level-set* representation.
Implicit surfaces via Support Vector Regression

- Generate training points along normals
- The number of points depends on the distance
- Interpolate using SVM-Regression
- Vary kernel width to get both accuracy and a sparse representation
Distance measure

Let $l_1, l_2$ be the signed distance functions of $\Gamma_1, \Gamma_2$. Possible distance functions

$$\mathcal{D}[\Gamma_1, \Gamma_2, \Phi] = \mathcal{D}[l_1, l_2, \Phi]$$

may include:

preserving signed distances

$$\int_\Omega (l_1(\Phi(x)) - l_2(x))^2 \, dx$$

higher order differential properties

$$\sum_i \int_\Omega [\nabla^i l_1(\Phi(x)) - \nabla^i l_2(x)]^2 \, dx$$
Transforms and Regularization

The transform $\Phi$ is chosen to be a linear function in an RKHS $\mathcal{H}$.

$$\Phi_d(x) = x_d + u_d(x) = x_d + \langle u_d, K_x \rangle_{\mathcal{H}}$$

- $K_x$ is the feature map corresponding to kernel $K$.
- Every dimension $d$ is treated independently.
- Regularization is performed by penalizing complex $u$, i.e.

$$\mathcal{R}[\Phi] = \mathcal{R}[u] = \sum_{d=1}^{D} \| u_d \|^2_{\mathcal{H}}$$
The optimization problem

- Manually clicked landmark points can make the registration problem easier.
- Let \((x_1, z_1), \ldots, (x_m, z_m)\) be \(m\) landmark pairs.
- The landmark costs are

\[
\sum_{i=1}^{m} \| \Phi(x_i) - z_i \|^2
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Full optimization problem

\[
\int_{\Omega} (l_1(\Phi(x)) - l_2(x))^2 \, dx + \lambda_l \sum_{i=1}^{m} \| \Phi(x_i) - z_i \|^2 + \lambda_r \sum_{d=1}^{D} \| \Phi_d \|^2_{\mathcal{H}}
\]
Solving the optimization problem

\[
\int_{\Omega} \left( l_1(x + u(x)) - l_2(x) \right)^2 \, dx + \lambda_l \sum_{i=1}^{m} \| x_i + u(x_i) - z_i \|^2 \\
+ \lambda_r \sum_{d=1}^{D} \| u_d \|^2_{H}
\]

- Sample at \( n \) points to get a tractable problem
- The *generalized representer theorem* states that the solution has the form

\[
u(\cdot) = \sum_{i=1}^{n} c_i K(x_i, \cdot)\]

- Solve for optimal \( c_i \) with gradient descent.
Solving the optimization problem

\[ \sum_{x \in \mathcal{X}} (l_1(x + u(x)) - l_2(x))^2 + \lambda_l \sum_{i=1}^{m} \| x_i + u(x_i) - z_i \|^2 + \lambda_r \sum_{d=1}^{D} \| u_d \|^2_{H} \]

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  \[ u(\cdot) = \sum_{i=1}^{n} c_i K(x_i, \cdot) \]
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Computing an initial solution

\[ \sum_{x \in X} (l_1(x + u(x)) - l_2(x))^2 + \lambda_l \sum_{i=1}^{m} \| x_i + u(x_i) - z_i \|^2 + \lambda_r \sum_{d=1}^{D} \| u_d \|_{\mathcal{H}}^2 \]

We get the \( d \) familiar problems

\[ \min_{u_d \in \mathcal{H}} \lambda_l \sum_{i=1}^{m} (x_{i,d} + u_d(x_i) - z_{i,d})^2 + \lambda_r \| u_d \|_{\mathcal{H}}^2 \]
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- Solve it using the \textit{classical representer theorem} and use the solution as an initial guess.
Some results

The results are taken from the original paper of Schölkopf et al.
• The registration problem is looked at from a machine learning perspective.
• The concepts are standard in registration, . . .
• . . ., but using RKHS and machine learning techniques is an interesting approach.
• The learning and registration problem are both instances of ill-posed interpolation problems.
  ⇒ it is not surprising, that the techniques are very similar.